From the continuous P_V to discrete Painlevé equations

2002 J. Phys. A: Math. Gen. 35 5943

(http://iopscience.iop.org/0305-4470/35/28/312)

Institute of Physics Publishing Journal of Physics A: Mathematical and General
J. Phys. A: Math. Gen. **35** (2002) 5943–5950 PII: S0305-4470(02)33093-2

From the continuous P_V to discrete Painlevé equations

T Tokihiro[1], **B Grammaticos**[2] **and A Ramani**[3]

[1] Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153, Japan
[2] GMPIB, Université Paris VII, Tour 24-14, 5^{e}étage, case 7021, 75251 Paris, France
[3] CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France

Received 28 January 2002, in final form 24 May 2002
Published 5 July 2002
Online at stacks.iop.org/JPhysA/35/5943

Abstract
We study the discrete transformations that are associated with the auto-Bäcklund of the (continuous) P_V equation. We show that several two-parameter discrete Painlevé equations can be obtained as contiguity relations of P_V. Among them we find the asymmetric d-P_{II} equation which is a well-known form of discrete P_{III}. The relation between the ternary P_I (previously obtained through the discrete dressing approach) and P_V is also established. A new discrete Painlevé equation is also derived.

PACS numbers: 02.30.−f, 02.30.Ik, 02.30.Gp

1. Introduction

One of the most interesting relations between continuous and discrete Painlevé equations (c- and d-$\mathbb{P}$) is that the contiguity relations of the former assume forms which can be found among the latter. The term 'contiguity relation' is used in analogy with the well-known property of special functions. Indeed, for the solutions of equations of the hypergeometric family there exist expressions that relate the solutions of the equation for the same value of the independent variable but different values of the parameters. A large class of d-$\mathbb{P}$ can be obtained in this way from the continuous Painlevé equations. One of the very first documented forms of discrete Painlevé equation, in fact, was the one derived by Jimbo and Miwa as the contiguity of P_{II} [1] (though it was not identified as a d-$\mathbb{P}$ at that time). The systematic derivation of the contiguity relations for c-$\mathbb{P}$ was already proposed in [2] as a means of obtaining new d-$\mathbb{P}$. The advantage of this construction was that it was based on the Schlesinger/auto-Bäcklund transformations of the various c-$\mathbb{P}$ and thus provided in a straightforward way the Lax pair of the d-$\mathbb{P}$. Let us show how this construction proceeds. We start with the Lax pair of a continuous Painlevé equation. It has the general form

$$\psi_\zeta = A\psi \tag{1.1a}$$

$$\psi_t = B\psi \tag{1.1b}$$

where ζ is the spectral parameter and A, B are matrices depending explicitly on ζ and the dependent as well as the independent variables w and t. The continuous Painlevé equation is obtained from the compatibility condition $\psi_{\zeta t} = \psi_{t\zeta}$ leading to

$$A_t - B_\zeta + AB - BA = 0. \tag{1.2}$$

In general, the Painlevé equation depends on parameters $(\alpha, \beta, \ldots)$ which are associated with the monodromy exponents θ_i appearing explicitly in the Lax pair. The Schlesinger transform relates two solutions ψ and ψ' of the isomonodromy problem for the equation at hand corresponding to different sets of parameters $(\alpha, \beta, \ldots)$ and $(\alpha', \beta', \ldots)$. The main characteristic of these transforms is that the monodromy exponents (at the singularities of the associated linear problem), related to the sets $(\alpha, \beta, \ldots)$ and $(\alpha', \beta', \ldots)$ differ by integers (or half-integers). The general form of a Schlesinger transformation is

$$\psi' = R\psi \tag{1.3}$$

where R is again a matrix depending on ζ, w, t and the monodromy exponents θ_i. The important remark is that $(1.1a)$ together with (1.3) constitute *the Lax pair of a discrete equation*. The latter is obtained from the compatibility conditions:

$$R_\zeta + RA - A'R = 0. \tag{1.4}$$

The analysis of the contiguity relations of the various c-$\mathbb{P}$ and their associated d-$\mathbb{P}$ has been presented in several papers. In [2], the so-called alternate d-P_{I} was derived from P_{II} (and it was given already in [1]). The discrete equations associated with the full P_{III} and the one-parameter P_{III} were first derived in [2] and studied in detail in [3, 4]. The asymmetric d-P_{I} related to P_{IV} was derived in [2] and studied in detail in [5]. Finally, the contiguity of P_{VI} was presented in [6]. Curiously, the discrete equations associated with P_{V} were never studied systematically. In [2], three such equations were derived, but two of them were in quite an unusual form; a clear sign that the derivation was not the proper one. In this paper we plan to remedy this by examining in detail the possible contiguity relations of P_{V}. We shall show that the three well-known discrete Painlevé equations can be described in such a way. In the process we will obtain a fourth, quite new, d-$\mathbb{P}$.

2. The auto-Bäcklund transformations of P_{V} and the elementary Miura

The Painlevé V equation we are going to work with is traditionally written as

$$v'' = \left(\frac{1}{2v} + \frac{1}{v-1}\right)v'^2 - \frac{v'}{z} + \frac{(v-1)^2}{z^2}\left(\alpha v + \frac{\beta}{v}\right) + \frac{\gamma v}{z} + \frac{\delta v(v+1)}{v-1}. \tag{2.1}$$

Before giving the auto-Bäcklund transformations we introduce a new, more convenient parametrization. First, through the appropriate scaling of the independent variable z we put $2\delta = -1$. We write $2\alpha = (n-p)^2$, $2\beta = -(n+p)^2$, $\gamma = -2q$. Furthermore, introducing the two independent signs $\epsilon = \pm 1$, $\eta = \pm 1$, we have $\sqrt{2\alpha} = \epsilon(n-p)$, $\sqrt{-2\beta} = \eta(n+p)$. Thus every instance of P_{V} is characterized by a triplet (n, p, q). We can now give the auto-Bäcklund [7]:

$$V = 1 - \frac{2zv}{zv' - \epsilon(n-p)v^2 + (\epsilon(n-p) - \eta(n+p) + z)v + \eta(n+p)} \tag{2.2}$$

which relate $v(n, p, q)$ to $V(N, P, Q)$ where (N, P, Q) are related to (n, p, q) through the following relations:

$$N = \sigma q$$
$$P = \begin{cases} \sigma\left(\frac{1}{2} - \eta n\right) & \text{if} \quad \epsilon\eta = 1 \\ \sigma\left(\frac{1}{2} - \eta p\right) & \text{if} \quad \epsilon\eta = -1 \end{cases} \tag{2.3}$$
$$Q = \begin{cases} -\eta p & \text{if} \quad \epsilon\eta = 1 \\ -\eta n & \text{if} \quad \epsilon\eta = -1 \end{cases}$$

where $\sigma = \pm 1$. From (2.2) it is clear that the auto-Bäcklund indeed introduces four transformations depending on the signs of ϵ, η. We introduce the notation $V_{\epsilon,\eta}$ in order to describe this dependence. With the appropriate combinations in (2.2) we find

$$\frac{z}{V_{1,\eta} - 1} - \frac{z}{V_{-1,\eta} - 1} = (n - p)(v - 1) \tag{2.4a}$$
$$\frac{z}{V_{\epsilon,1} - 1} - \frac{z}{V_{\epsilon,-1} - 1} = (n + p)\left(1 - \frac{1}{v}\right). \tag{2.4b}$$

It is more convenient now to introduce the variable $w = \frac{v+1}{v-1}$ (and similarly $W = \frac{V+1}{V-1}$) whereupon we find

$$W_{1,\eta} - W_{-1,\eta} = \frac{4}{z}\frac{n - p}{w - 1} \tag{2.5a}$$
$$W_{\epsilon,1} - W_{\epsilon,-1} = \frac{4}{z}\frac{n + p}{w + 1}. \tag{2.5b}$$

In order to proceed, we shall need one other property of the solutions of P_V. If v is a solution of P_V with parameters $(\alpha, \beta, \gamma, \delta)$, then $1/v$ is also a solution with parameters $(-\beta, -\alpha, -\gamma, \delta)$. In order to transcribe this into our parametrization we first note that $v \to 1/v$ corresponds to $w \to -w$. Next, from the form of P_V it is clear that the equation is invariant under a simultaneous change of sign of n and p as well as under the exchange of n and p. Thus w can be characterized by any of the following four sets of parameters: $(n, p, q), (p, n, q), (-n, -p, q), (-p, -n, q)$. Similarly $-w$ is associated with the four sets $(n, -p, -q), (p, -n, -q), (-n, p, -q), (-p, n, -q)$, i.e. $w \to -w$ changes the sign of the third parameter q and that of *either n or p*.

Next we introduce the shorthand notation:

$$\begin{aligned} \bar{w} &= W_{-1,-1} && \text{with parameters} \quad \left(q, n + \tfrac{1}{2}, p\right) \\ \underline{w} &= -W_{1,1} && \text{with parameters} \quad \left(q, n - \tfrac{1}{2}, p\right) \\ \hat{w} &= W_{1,-1} && \text{with parameters} \quad \left(q, p + \tfrac{1}{2}, n\right) \\ \underset{\wedge}{w} &= -W_{-1,1} && \text{with parameters} \quad \left(q, p - \tfrac{1}{2}, n\right). \end{aligned}$$

We readily see that the bar ($\bar{\ }$) denotes evolution in the n direction while the hat ($\hat{\ }$) is associated with the p direction. We can now rewrite (2.5) as

$$\hat{w} + \bar{w} = \frac{4}{z}\frac{n - p}{w - 1} = \underset{\wedge}{w} - \underline{w} \tag{2.6a}$$

and similarly

$$\underline{w} + \hat{w} = -\frac{4}{z}\frac{n + p}{w + 1} = \bar{w} + \underset{\wedge}{w}. \tag{2.6b}$$

Equations (2.6*a*) and (2.6*b*) constitute the elementary discrete Miuras.

3. Derivation of the discrete Painlevé equations associated with P_V

The Miura transformations (2.6) constitute the building blocks for the construction of the discrete Painlevé equations from the auto-Bäcklund of P_V.

The first discrete equation we are going to derive is known as asymmetric d-P_{II}, which as shown in [8] (where we also obtained its Lax pair) is a discrete form of P_{III}. Subtracting the rhs of (2.6*b*) from that of (2.6*a*), we find

$$\bar{w} + \underset{\sim}{w} = -\frac{4}{z}\left(\frac{n+p}{w+1} + \frac{n-p}{w-1}\right) = -\frac{8}{z}\frac{nw-p}{w^2-1}. \tag{3.1}$$

This is not yet the equation we seek because when we move from w to $\bar{w}$ (or $\underset{\sim}{w}$) n increases (decreases) by $\frac{1}{2}$ but simultaneously p and q are permuted. This means that when we consider the equation relating w, $\bar{w}$ and $\bar{\bar{w}}$, we find

$$w + \bar{\bar{w}} = -\frac{8}{z}\frac{\left(n+\frac{1}{2}\right)\bar{w} - q}{\bar{w}^2 - 1}. \tag{3.2}$$

Of course, at the level of $\bar{w}$, $\bar{\bar{w}}$ and $\bar{\bar{\bar{w}}}$ we will again find an equation such as (3.1) with $(n+1)$ instead of n. Thus the equation related to this evolution has an even/odd dependence in the additive constant in the numerator as expected for the asymmetric d-P_{II}.

In [9] we studied this equation from the point of view of the self-duality property. Self-duality in this context means that the same discrete equation describes the evolution along the independent variable and also along the parameter changes due to Schlesinger/auto-Bäcklund transformations. The self-duality of asymmetric d-P_{II} is very easy to understand in the frame of the present derivation. Indeed, adding the rhs of (2.6*a*) to the lhs of (2.6*b*), we obtain

$$\hat{w} + \underset{\wedge}{w} = -\frac{8}{z}\frac{pw-n}{w^2-1}. \tag{3.3}$$

The hat evolution corresponds to a move from p to $p \pm \frac{1}{2}$ while n and $q - \frac{1}{2}$ are permuted. In a perfect parallel to (3.2) we have also

$$w + \hat{\hat{w}} = -\frac{8}{z}\frac{\left(p+\frac{1}{2}\right)\hat{w} - q}{\hat{w}^2 - 1}. \tag{3.4}$$

Similarly, we could have introduced an evolution along the q direction through $\underset{\sim}{w}$, $\bar{w}$, $\underset{\wedge}{w}$ or $\hat{w}$ but not through w: through any point one can evolve in two directions only, since there are only four auto-Bäcklund transformations of the form (2.2). The result is again an asymmetric d-P_{II} equation in which q plays the role of the independent variable while n and p enter as the even/odd dependent constants.

We turn now to another discrete equation which can be obtained from the same Miura (2.6). From (2.6*b*) we have

$$\underset{\sim}{w} + \hat{w} = -\frac{4}{z}\frac{n+p}{w+1}. \tag{3.5}$$

The corresponding parameters of $\underset{\sim}{w}$, w and $\hat{w}$ are $\left(q, n-\frac{1}{2}, p\right)$, (n, p, q) and $\left(p+\frac{1}{2}, q, n\right)$, respectively. We note that going from $\underset{\sim}{w}$ to w to $\hat{w}$ the parameters of the solution change according to the following pattern: $(\phi, \psi, \theta) \to \left(\psi + \frac{1}{2}, \theta, \phi\right)$ (and at the numerator of the rhs we have the quantity $\phi + \psi$ corresponding to the 'middle variable', w in this case, so $\phi + \psi = n + p$). Thus schematically we have three different equations along the variables associated with the triplets:

$$(\theta, \underset{\sim}{\phi}, \psi), (\phi, \psi, \theta), (\tilde{\psi}, \theta, \phi)$$
$$(\phi, \psi, \theta), (\tilde{\psi}, \theta, \phi), (\tilde{\theta}, \phi, \tilde{\psi})$$
$$(\tilde{\psi}, \theta, \phi), (\tilde{\theta}, \phi, \tilde{\psi}), (\tilde{\phi}, \tilde{\psi}, \tilde{\theta})$$

where the tilde (˜) means that the particular parameter is incremented by $\frac{1}{2}$.

The corresponding numerators of the rhs are $\phi + \psi$, $\tilde{\psi} + \theta$ and $\tilde{\theta} + \phi$, respectively. The next equation would involve the triplets $(\tilde{\theta}, \phi, \tilde{\psi})$, $(\tilde{\phi}, \tilde{\psi}, \tilde{\theta})$ and $(\tilde{\tilde{\psi}}, \tilde{\theta}, \tilde{\phi})$ with the numerator $\tilde{\phi} + \tilde{\psi} = \phi + \psi + 1$. It is just the upshift of the first equation. Thus the Miura (3.5) introduces an equation with ternary symmetry which can be written schematically as

$$u_{m-1} + u_{m+1} = 2 - \frac{4}{z}\frac{k_m}{u_m} \tag{3.6}$$

with the change of variable $u = w + 1$. The index m denotes the evolutions from $\underset{\sim}{w}$ to w to $\hat{w}$, etc and from the above analysis one sees that $k_{m+3} = k_m + 1$, i.e. k grows linearly with a superimposed period-3 variation (as k_{3k} ignores θ, k_{3k+1} ignores ϕ and k_{3k+2} ignores ψ, if we assume that the index of the variable w, say, is an integer multiple of 3).

This equation is well known. In [10], we derived it from the discrete dressing transformation. While its symmetric (without the ternary dependence) form is a discrete P_I [11], the full equation was shown to be a discrete form of P_{IV}.

In order to derive the next equation we start from the Miura (3.5) and rewrite it as

$$(\underset{\sim}{w} - 1)(w + 1) + (\hat{w} + 1)(w + 1) = -\frac{2}{z}\left(2n - \frac{1}{2} + 2p + \frac{1}{2}\right). \tag{3.7}$$

Next we introduce the auxiliary quantity χ and split (3.7) into two equations:

$$(\underset{\sim}{w} - 1)(w + 1) = -\frac{2}{z}\left(2n - \frac{1}{2} + \chi\right) \tag{3.8a}$$

$$(\hat{w} + 1)(w + 1) = -\frac{2}{z}\left(2p + \frac{1}{2} - \chi\right). \tag{3.8b}$$

Similarly, consider the Miura

$$w + \breve{\hat{w}} = -\frac{4}{z}\frac{p + q + \frac{1}{2}}{\hat{w} + 1} \tag{3.9}$$

where the breve (˘) denotes the evolution in the q direction (which, as we said above, can go through $\hat{w}$) while the numerator $p + q + \frac{1}{2}$ is indeed the sum of the parameters along the directions p and q at $\hat{w}$. We can rewrite it as

$$(w + 1)(\hat{w} + 1) + (\breve{\hat{w}} - 1)(\hat{w} + 1) = -\frac{2}{z}\left(2p + \frac{1}{2} + 2q + \frac{1}{2}\right). \tag{3.10}$$

Using (3.8b) one sees that

$$(\breve{\hat{w}} - 1)(\hat{w} + 1) = -\frac{2}{z}\left(2q + \frac{1}{2} + \chi\right). \tag{3.11}$$

Combining (3.8a), (3.8b) and (3.11) we find

$$(\breve{\hat{w}} - 1)(\underset{\sim}{w} - 1) = \frac{2}{z}\frac{\left(\chi + 2q + \frac{1}{2}\right)\left(\chi + 2n - \frac{1}{2}\right)}{\chi - (2p + 1/2)}. \tag{3.12}$$

Next we start from (3.8*a*) and upshift it in all three directions obtaining

$$(\breve{\hat{w}}-1)(\bar{\breve{\hat{w}}}+1)=-\frac{2}{z}\left(2n+\frac{1}{2}+\bar{\breve{\hat{\chi}}}\right). \tag{3.13}$$

Adding (3.13) to (3.11) we find

$$(\breve{\hat{w}}-1)(\bar{\breve{\hat{w}}}+\hat{w}+2)=-\frac{2}{z}(2n+2q+1+\chi+\bar{\breve{\hat{\chi}}}). \tag{3.14}$$

We also have one more Miura

$$\hat{w}+\bar{\breve{\hat{w}}}=-\frac{4}{z}\frac{n+q+\frac{1}{2}}{\breve{\hat{w}}+1} \tag{3.15}$$

(note that $\breve{\hat{w}}$ is the variable associated with $(q+\frac{1}{2}, n, p+\frac{1}{2})$, upshifted from $\underline{w}$ at $(q, n-\frac{1}{2}, p)$ in all three directions, and the corresponding numerator is indeed $n+q+\frac{1}{2}$), so this leads to

$$(\breve{\hat{w}}-1)\left(2-\frac{4}{z}\frac{n+q+\frac{1}{2}}{\breve{\hat{w}}+1}\right)=-\frac{2}{z}(2n+2q+1+\chi+\bar{\breve{\hat{\chi}}}) \tag{3.16}$$

which can be rewritten as

$$\chi+\bar{\breve{\hat{\chi}}}=-\frac{4(n+q)+2}{\breve{\hat{w}}+1}-z(\breve{\hat{w}}-1). \tag{3.17}$$

Equations (3.12) and (3.17) can be written in a simpler way if we introduce $\omega = w-1$. We have

$$\breve{\hat{\omega}}\underline{\omega}=\frac{2}{z}\frac{\left(\chi+2q+\frac{1}{2}\right)\left(\chi+2n-\frac{1}{2}\right)}{\chi-2p-1/2} \tag{3.18a}$$

$$\chi+\bar{\breve{\hat{\chi}}}=-\frac{4(n+q)+2}{\breve{\hat{\omega}}+2}-z\breve{\hat{\omega}}. \tag{3.18b}$$

Note that $w=\omega+1$ is a solution of the continuous P_V (though in a form slightly different from the canonical one) but χ does not: it satisfies some second-order equation which is of degree 4 in χ'' and is much too cumbersome to be written here. Equation (3.18) was first derived in [2], from the Schesinger transformations of P_V. It was obtained again in [12] from the degeneration pattern of the asymmetric q-P_{III} (discrete P_{VI}) equation [13] and where its Lax pair has been derived. This equation has been shown to be another discrete form of P_{IV}.

We now turn to yet another discrete equation which can be derived from P_V. Our starting point is again the lhs of the Miura (2.6*b*):

$$\underline{w}+\hat{w}=-\frac{4}{z}\frac{n+p}{w+1}$$

and the discrete relations relating $\underline{\underline{w}}$, $\underline{w}$, w:

$$\underline{\underline{w}}+w=-\frac{8}{z}\frac{\left(n-\frac{1}{2}\right)\underline{w}-q}{\underline{w}^2-1} \tag{3.19a}$$

and w, $\hat{w}$ and $\hat{\hat{w}}$:

$$w+\hat{\hat{w}}=-\frac{8}{z}\frac{\left(p+\frac{1}{2}\right)\hat{w}-q}{\hat{w}^2-1}. \tag{3.19b}$$

Next we consider the discrete Miura equation relating $\underset{\wedge}{\underline{\underline{w}}}$, $\underline{\underline{w}}$, $\underline{w}$ and $\hat{w}$, $\hat{\hat{w}}$ and $\bar{\hat{\hat{w}}}$, respectively:

$$\underset{\wedge}{\underline{\underline{w}}}+\underline{w}=-\frac{4}{z}\frac{n+p-1}{\underline{\underline{w}}+1} \qquad \hat{w}+\bar{\hat{\hat{w}}}=-\frac{4}{z}\frac{n+p+1}{\hat{\hat{w}}+1}. \tag{3.19c, d}$$

Now we introduce the notation check $(\check{\ })$ in order to indicate the shift in both n and p, i.e. $(\check{\ }) = (\hat{\bar{\ }})$, $\underset{\vee}{x}$, x, $\check{x}$ for the variables $\underline{\underline{w}}$, $\underline{w}$ and $\hat{w}$ and the variables $\underline{w}$, w and $\hat{\hat{w}}$ will be called $\underset{\vee}{y}$, y, $\check{y}$. This is a slight misnomer, as the motion from $\underset{\vee}{y}$ to y and from y to $\check{y}$ are $(\bar{\bar{\ }})$ and $(\hat{\hat{\ }})$, respectively, instead of $(\hat{\bar{\ }})$ each but it 'averages' out correctly in the end. We now have the following succession of variables: $\ldots, \underset{\vee}{y}, x, y, \check{x}, \check{y}, \ldots$. The discrete Painlevé equation then assumes the form

$$x + \check{x} = -\frac{4}{z}\frac{k + \check{k}}{y + 1} \tag{3.20a}$$

$$\underset{\vee}{y} + y = -\frac{8}{z}\frac{kx - q}{x^2 - 1} \tag{3.20b}$$

where k takes the current value of n and p alternatively (so $k + \check{k}$ always take the local value of $n + p$) and q is constant along this evolution. Indeed, (3.19a) and (3.19b) are two instances of (3.20b) while the lhs of (2.6b), (3.19c) and (3.19d) are three instances of (3.20a). We readily note that k in (3.20a) varies depending on the position, i.e. it has an even/odd dependence. We can rewrite equation (3.20), by eliminating y, as a single equation for x:

$$\frac{k + \check{k}}{x + \check{x}} + \frac{k + \underset{\vee}{k}}{x + \underset{\vee}{x}} - 4 = \frac{2kx - 2q}{x^2 - 1}. \tag{3.21}$$

As a matter of fact in equation (3.21) k has an even/odd dependence (which introduces an extra parameter). Let us point out here that a study of (3.20) using the singularity confinement integrability criterion predicts precisely this even/odd dependence to be compatible with integrability.

In the derivation presented above, the equations are given in the forms they occur naturally with just a minimum of variable transformations. Here we would like to summarize our findings and present these equations in the usual form. All equations will be written for a single variable which we will write x_m, the independent variable being introduced $z_m = \delta(m - m_0)$. The step δ is related to the independent variable of the continuous P_V: $\delta \propto 1/z$. The even/odd or ternary freedom will be introduced through the appropriate phases and, of course, the corresponding discrete equation could have been written as a system for more than one dependent variable. Thus we have, for the asymmetric d-P_{II} (3.2), the form

$$x_{m+1} + x_{m-1} = \frac{z_m x_m + a + b(-1)^m}{x_m^2 - 1}. \tag{3.22}$$

For the ternary d-P_I (3.6) we find

$$x_{m+1} + x_{m-1} = 1 + \frac{z_m + aj^m + bj^{2m}}{x_m} \tag{3.23}$$

where j is a (complex) cubic root of unity. Equation (3.18) can be rewritten as

$$x_{m-1} + x_m = \frac{1}{y_m} + \frac{z_m + a}{1 - y_m} \tag{3.24a}$$

$$y_m y_{m+1} = \frac{x_m - z_m}{x_m^2 - b^2} \tag{3.24b}$$

where $y = -2/\omega$. Finally, equation (3.21) can be written as

$$\frac{z_{m-1} + z_m}{x_{m-1} + x_m} + \frac{z_{m+1} + z_m}{x_{m+1} + x_m} = 1 + \frac{(2z_m + a(-1)^m)x_m + b}{x_m^2 - 1} \tag{3.25}$$

where two parameters now appear, as expected.

4. Conclusion

In this paper we have presented a systematic construction of the discrete equations one can obtain as contiguity relations of the continuous Painlevé V. We have shown how one can relate some well-known discrete Painlevé equations, namely the asymmetric d-P_{II} (discrete P_{III}) and the ternary d-P_I (discrete P_{IV}), to P_V. Equation (3.18) was already obtained in [2] where discrete equations were obtained from the Schlesinger transformations of P_V. However, the absence of the guide provided by the geometry of the system, namely the affine Weyl group $A_3^{(1)}$ which describes the transformations of the parameters of P_V, resulted in the fact that all the other equations obtained from P_V in [2] were given in a highly unusual, rather intractable, form. Here the proper use of the geometric guide allowed us not only to recover the previously known d-$\mathbb{P}$ and relate them to P_V but also to discover a d-$\mathbb{P}$ which has never been encountered before. We must point out here that, given the richness of the space of $A_3^{(1)}$, there certainly exist discrete equations corresponding to more complicated paths. In fact, any nonclosed pattern, periodically repeated, would lead to some discrete equation, with higher and higher periodicity.

As we have shown in this paper, the construction of contiguity relations of c-$\mathbb{P}$ starting from the auto-Bäcklund transformations is a fruitful approach. One could thus wonder whether the same procedure could be applied to the auto-Bäcklund's *discrete* Painlevé equations and thus continue this construction iteratively until all the parameters are exhausted. It has turned out that this is not possible. The geometry of the underlying transformations is such that all difference Painlevé equations are self-dual, i.e. the contiguity of a given d-$\mathbb{P}$ is a d-$\mathbb{P}$ of the *same* form. In the case of q-discrete Painlevé equations the same holds true for the vast majority of equations (although some q-$\mathbb{P}$ exist which are *not* self-dual). What is more intriguing is that there exist difference Painlevé equations which have four or more parameters and thus cannot be contiguity relations of the known continuous Painlevé equations. The investigation of the exact nature of these equations and the derivation of their Lax pair constitute a genuine challenge.

References

[1] Jimbo M and Miwa T 1981 *Physica* D **2** 407
[2] Fokas A S, Grammaticos B and Ramani A 1993 *J. Math. Anal. Appl.* **180** 342
[3] Nijhoff F, Satsuma J, Kajiwara K, Grammaticos B and Ramani A 1996 *Inverse Problems* **12** 697
[4] Ramani A, Grammaticos B, Tamizhmani T and Tamizhmani K M 2000 *J. Phys. A: Math. Gen.* **33** 579
[5] Grammaticos B and Ramani A 1998 *J. Phys. A: Math. Gen.* **31** 5787
[6] Nijhoff F, Ramani A, Grammaticos B and Ohta Y 2000 *Stud. Appl. Math.* **106** 261
[7] Fokas A S and Ablowitz M J 1982 *J. Math. Phys.* **23** 2033
[8] Grammaticos B, Nijhoff F W, Papageorgiou V, Ramani A and Satsuma J 1994 *Phys. Lett.* A **185** 446
[9] Ramani A and Grammaticos B 1996 The grand scheme for discrete Painlevé equations *Lecture at the Toda Symposium* (1996)
[10] Grammaticos B, Ramani A and Papageorgiou V 1997 *Phys. Lett.* A **235** 475
[11] Ramani A and Grammaticos B 1996 *Physica* A **228** 160
[12] Grammaticos B, Ohta Y, Ramani A and Sakai H 1998 *J. Phys. A: Math. Gen.* **31** 3545
[13] Jimbo M and Sakai H 1996 *Lett. Math. Phys.* **38** 145